Optimal Criterion

Payoff Matrix

Stimulus \ Response	Yes	No
S+N	V_{S+N}^{Yes}	V_{S+N}^{No}
N	V_{N}^{Yes}	V_{N}^{No}

Optimal Criterion

Stimulus \ Response	Yes	No
S+N	V_{S+N}^{Yes}	V_{S+N}^{No}
N	V_{N}^{Yes}	V_{N}^{No}

$$E(Yes \mid x) = V_{S+N}^{Yes} p(S+N \mid x) + V_{N}^{Yes} p(N \mid x)$$
$$E(No \mid x) = V_{S+N}^{No} p(S+N \mid x) + V_{N}^{No} p(N \mid x)$$

Say yes if $E(Yes \mid x) \geq E(No \mid x)$

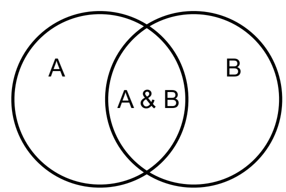
Optimal Criterion

$$E(Yes \mid x) = V_{S+N}^{Yes} p(S+N \mid x) + V_{N}^{Yes} p(N \mid x)$$
$$E(No \mid x) = V_{S+N}^{No} p(S+N \mid x) + V_{N}^{No} p(N \mid x)$$

Say yes if $E(Yes \mid x) \geq E(No \mid x)$

Say yes if $\dfrac{p(S+N \mid x)}{p(N \mid x)} \geq \dfrac{V_N^{No} - V_N^{Yes}}{V_{S+N}^{Yes} - V_{S+N}^{No}} = \dfrac{V(\text{Correct} \mid N)}{V(\text{Correct} \mid S+N)}$

Aside: Bayes' Rule



$p(A \mid B) =$ probability of A given that B is asserted to be true $= \dfrac{p(A \,\&\, B)}{p(B)}$

$$p(A \,\&\, B) = p(B)p(A \mid B)$$
$$= p(A)p(B \mid A)$$
$$\Rightarrow p(A \mid B) = \frac{p(B \mid A)p(A)}{p(B)}$$

Apply Bayes' Rule

$$p(S+N \mid x) = \frac{p(x \mid S+N)p(S+N)}{p(x)}$$

(Posterior: $p(S+N \mid x)$; Likelihood: $p(x \mid S+N)$; Prior: $p(S+N)$; Nuisance normalizing term: $p(x)$)

$p(N \mid x) = \dfrac{p(x \mid N)p(N)}{p(x)}$, hence

$$\frac{p(S+N \mid x)}{p(N \mid x)} = \left(\frac{p(x \mid S+N)}{p(x \mid N)}\right)\left(\frac{p(S+N)}{p(N)}\right)$$

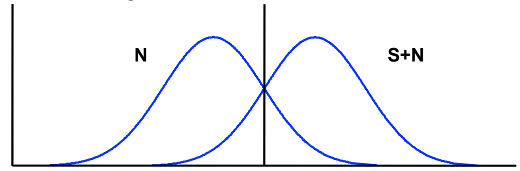
(Posterior odds = Likelihood ratio × Prior odds)

Optimal Criterion

Say yes if $\dfrac{p(S+N \mid x)}{p(N \mid x)} \geq \dfrac{V(\text{Correct} \mid N)}{V(\text{Correct} \mid S+N)}$

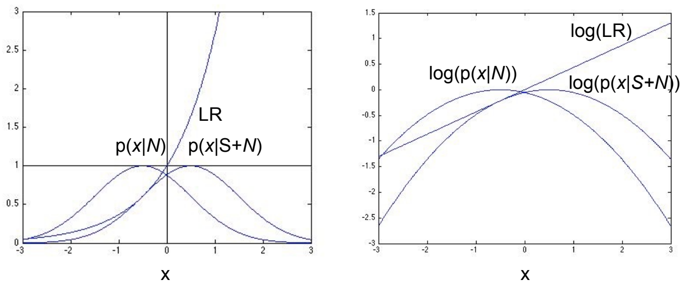
i.e., if $\dfrac{p(x \mid S+N)}{p(x \mid N)} \geq \dfrac{p(N)}{p(S+N)} \dfrac{V(\text{Correct} \mid N)}{V(\text{Correct} \mid S+N)} = \beta$

Example, if equal priors and equal payoffs, say yes if the likelihood ratio is greater than one:

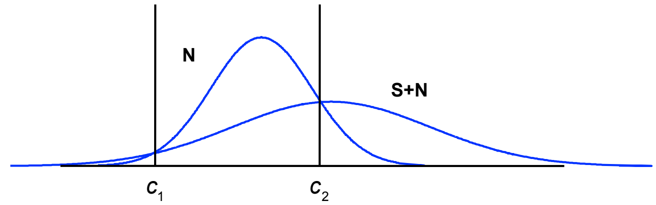


Likelihood Ratio

Optimal binary decisions are a function of the likelihood ratio (it is a sufficient statistic, i.e., no more information is needed or useful).



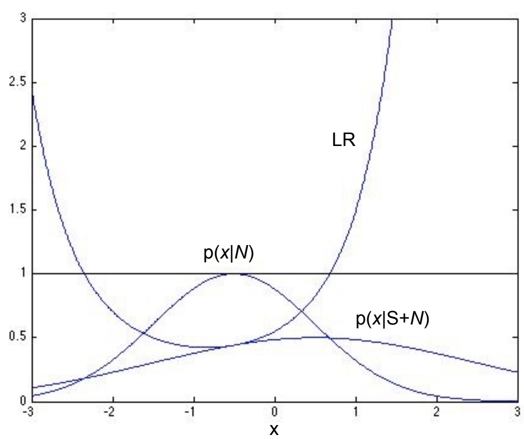
Gaussian Unequal Variance



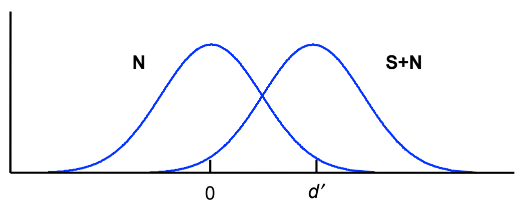
SDT (suboptimal): Say yes if $x \geq c_2$

Optimal strategy: Say yes if $x \geq c_2$ or $x \leq c_1$

Gaussian Unequal Variance



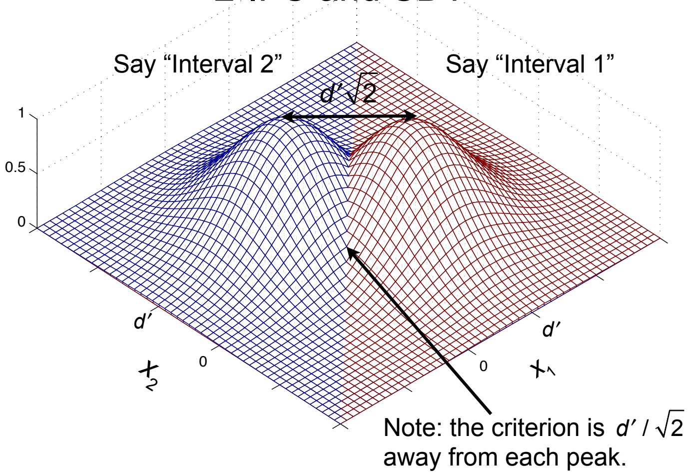
2-IFC and SDT



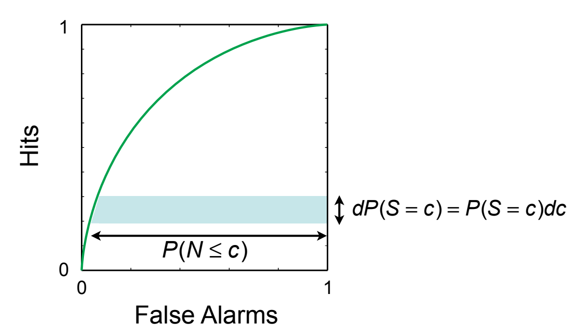
$$\begin{aligned} P(\text{Correct}) &= P(N(d',1) > N(0,1)) \\ &= P(N(d',1) - N(0,1) > 0) \\ &= P(N(d',\sqrt{2}) > 0) \\ &= P(N(d'/\sqrt{2},1) > 0) \\ &= P(N(0,1) < d'/\sqrt{2}) \end{aligned}$$

Where $N(\mu,\sigma)$ means a random variable with mean μ and SD σ.

2-IFC and SDT



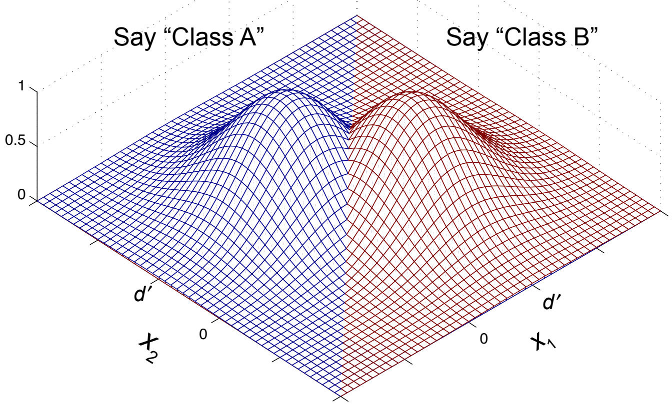
2-IFC and the Yes-No ROC



$$P(\text{Correct in 2-IFC}) = \int P(S=c)P(N \leq c)dc$$

$=$ Area under the ROC

General Gaussian Classification



General Gaussian Classification

The *d*-dimensional Gaussian distribution:

$$p(\vec{x}) = \frac{1}{(2\pi)^{d/2}\left|\Sigma\right|^{1/2}} \exp\left[-\frac{1}{2}(\vec{x}-\vec{\mu})^T \Sigma^{-1}(\vec{x}-\vec{\mu})\right]$$

$\Sigma =$ the covariance matrix

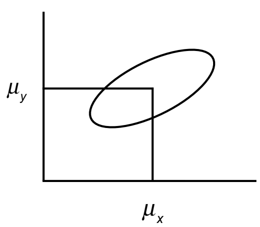
General Gaussian Classification

In 2 dimensions:

Mahalanobis distance

$$p(\vec{x}) = \frac{1}{2\pi\left|\Sigma\right|^{1/2}} \exp\left[-\frac{1}{2}(\vec{x}-\vec{\mu})^T \Sigma^{-1}(\vec{x}-\vec{\mu})\right]$$

$$\Sigma = \begin{bmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{bmatrix}$$



General Gaussian Classification

Say “Category A” if

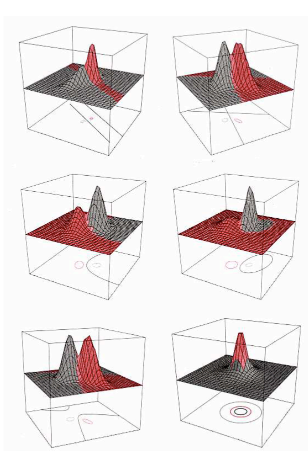
$$LR(\vec{x}) = \frac{\dfrac{1}{(2\pi)^{d/2}\left|\Sigma_A\right|^{1/2}} \exp\left[-\frac{1}{2}(\vec{x}-\vec{\mu}_A)^T \Sigma_A^{-1}(\vec{x}-\vec{\mu}_A)\right]}{\dfrac{1}{(2\pi)^{d/2}\left|\Sigma_B\right|^{1/2}} \exp\left[-\frac{1}{2}(\vec{x}-\vec{\mu}_B)^T \Sigma_B^{-1}(\vec{x}-\vec{\mu}_B)\right]} > 1$$

Take log and simplify. Say “Category A” if

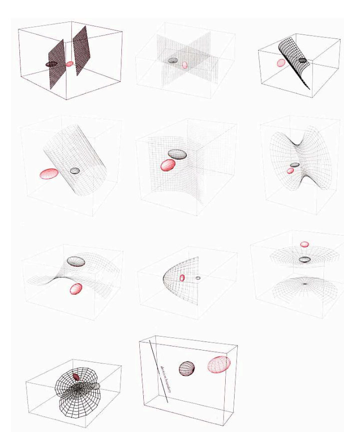
$$(\vec{x}-\vec{\mu}_B)^T \Sigma_B^{-1}(\vec{x}-\vec{\mu}_B) - (\vec{x}-\vec{\mu}_A)^T \Sigma_A^{-1}(\vec{x}-\vec{\mu}_A)$$
$$= MD(\vec{x},\vec{\mu}_B) - MD(\vec{x},\vec{\mu}_A) > C$$

which is a quadratic in $\vec{x}$.

General Gaussian Classification



General Gaussian Classification

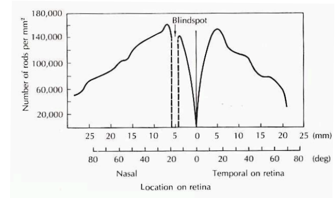


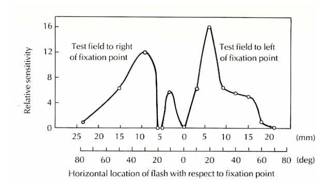
Application: Absolute Threshold

Question: What is the minimal number of photons required for a stimulus to be visible under the best of viewing conditions?

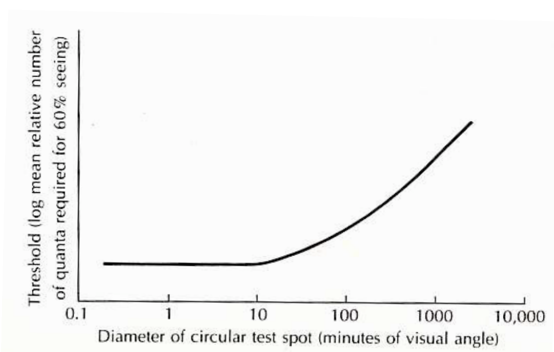
Hecht, Schlaer & Pirenne (1942)

Sensitivity by Eccentricity

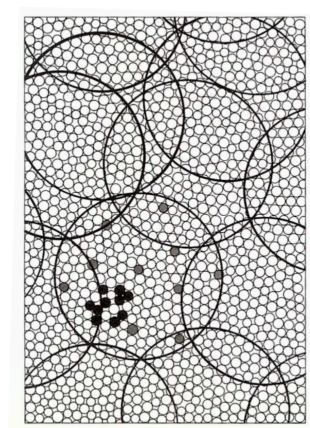




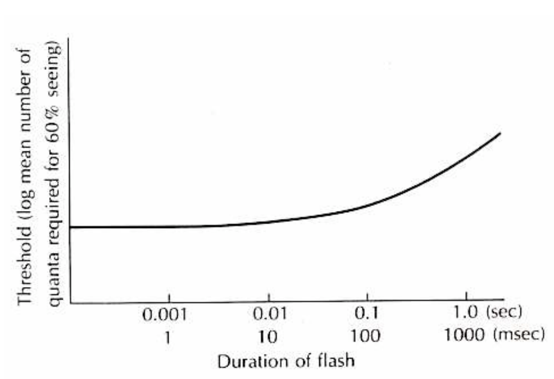
Spatial Summation



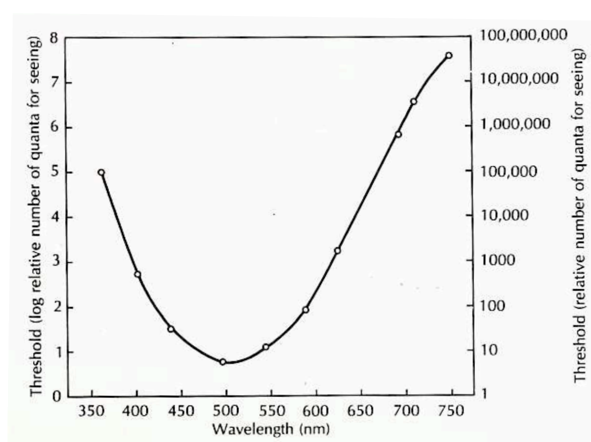
Spatial Summation



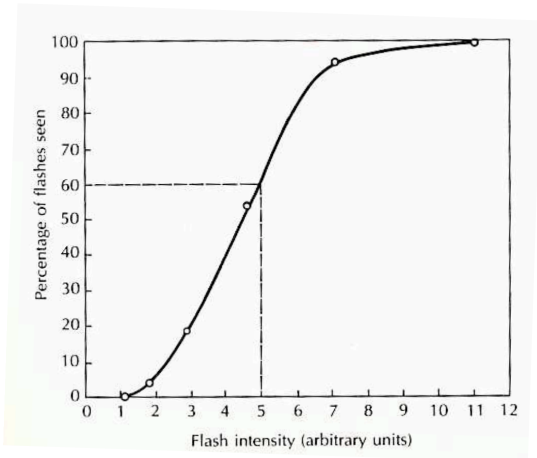
Temporal Summation



Choice of Wavelength



The Data



Poisson Distribution

Counting distribution for processes consisting of “events” that occur randomly at a fixed rate **λ** events/sec.

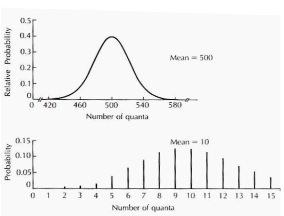
The time until the next event occurs follows an exponential distribution: $p(t) = \lambda e^{-\lambda t}$

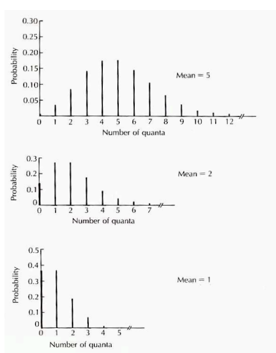
Since **λ** is the rate, then over a period of **τ** seconds, the expected count is **λ τ**. The distribution of event count is Poisson:

$$p(k) = \frac{e^{-\lambda\tau}(\lambda\tau)^k}{k!}$$

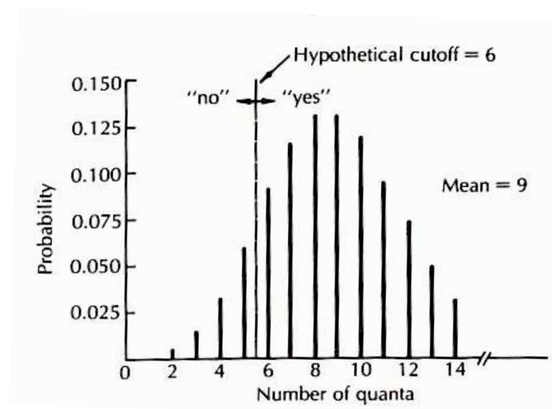
Stimulus Variability (Poisson)

$$p(k) = \frac{e^{-\lambda\tau}(\lambda\tau)^k}{k!}$$

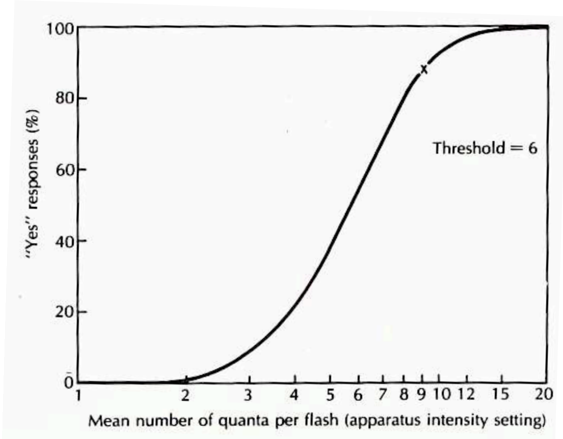




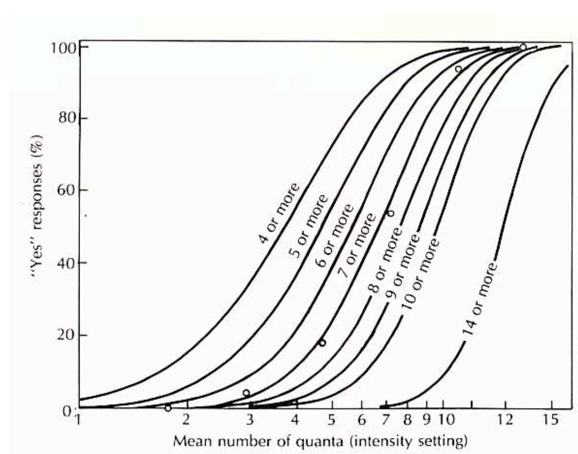
Model with no Subject Variability



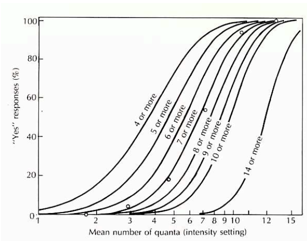
Model with no Subject Variability

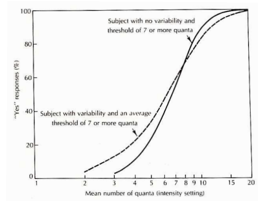


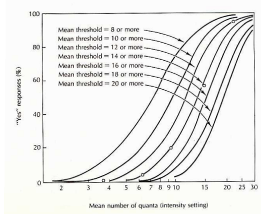
Model with no Subject Variability



Model with Subject Variability







Bayes classifier: ideal observer

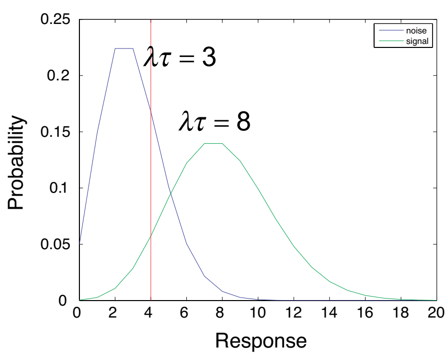
Question: What is the minimal contrast required to carry out various detection and discrimination tasks?

Geisler (1989)
See also Wandell Appendix 3

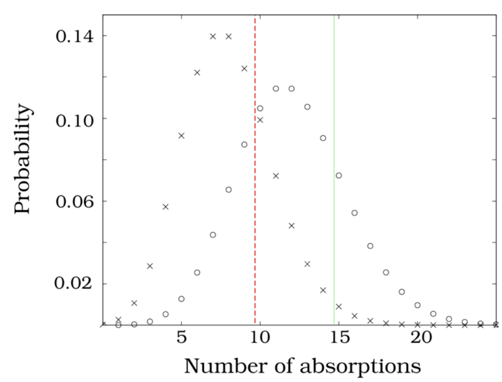
Examples:
- detecting a light
- distinguishing two spatial patterns

Poisson Distribution SDT

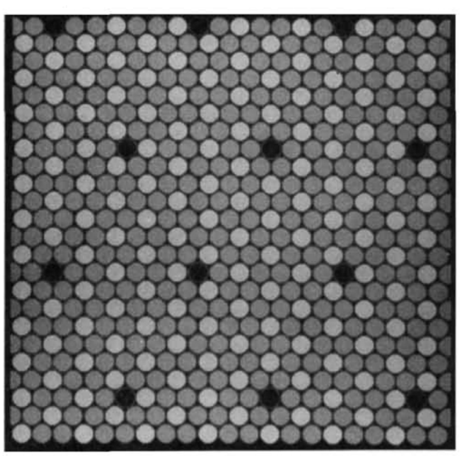
$$p(k) = \frac{e^{-\lambda\tau}(\lambda\tau)^k}{k!} \qquad LR(k) \propto \left(\frac{\lambda_{S+N}}{\lambda_N}\right)^k$$



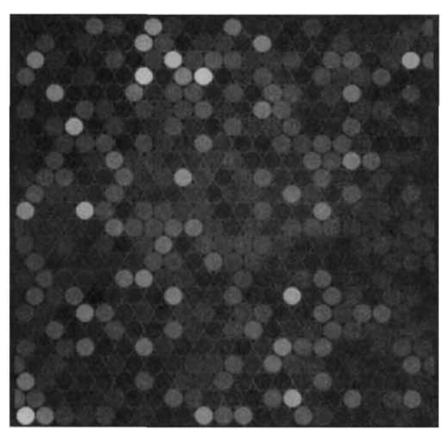
Optimal Criterion



Idealized Receptor Lattice



Poisson Statistics



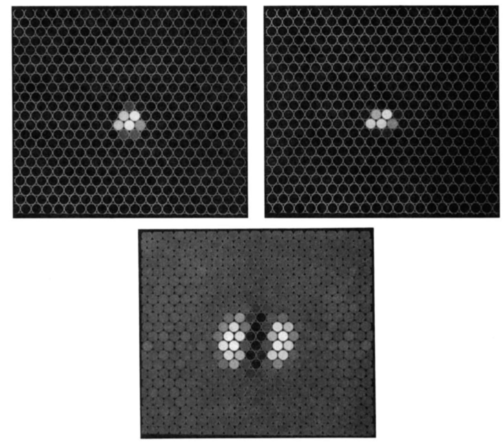
Ideal Discriminator

Given two known possible targets A and B with expected photon absorptions a_i and b_i and actual photon catches Z_i in receptor *i,* respectively, calculate the likelihood ratio $p(Z_i \mid a_i)/p(Z_i \mid b_i)$, take the log, do some algebra and discover the following quantity is monotonic in likelihood ratio:

$$Z = \sum_i Z_i \ln\left(a_i / b_i\right)$$

Decisions are thus based on an “ideal receptive field.”

Example: 2-Point Resolution



Bayes classifier for spatial discrimination